Sums of squares of binomial coefficients, with applications to Picard-Fuchs equations

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Abstract

For an arbitrary fixed positive integer N, we give a recurrence relation (12) for the sequence $a_n^N := \sum \left(\frac{n!}{p_1!p_2!\cdots p_N!}\right)^2$, where the sum runs over sets of non-negative integers $p_1,\ldots,p_N$ summing to n. For the more general sequence $a_n^{\alpha_1,\ldots,\alpha_N} := \sum \left(\prod_{i=1}^N \alpha_i^{p_i}\right)\left(\frac{n!}{p_1!p_2!\cdots p_N!}\right)^2$, for any fixed $\alpha_1,\ldots\alpha_N \in \mathbb{C}$, we give an algorithm for determining a recurrence relation. We show how to apply the latter procedure to obtain general recurrence relations ((24) and (32)) for the cases $N=2$ and 3. As a geometric application, in Section 5 we give the Picard-Fuchs equation (43) for the families of elliptic curves corresponding to the $N=3$ case.

The main idea is to introduce auxiliary sequences (7) or (18), generalising and systematising the method used in [V1] for the case $N=3, \alpha_1,\alpha_2,\alpha_3=1$. This approach seems to be similar to the method used by Cusick [C] in finding recurrences for sums of powers of binomials. We briefly discuss the application to this case in Section 6.

1 Introduction

In this paper we study sequences $\{a_n^{\boldsymbol{\alpha}}\}_n$ where

$$a_n^{\{\alpha_1,\ldots,\alpha_N\}} := \sum_{p_1+\cdots+p_N=n} \left(\prod_{i=1}^N \alpha_i^{p_i}\right)\left(\frac{n!}{p_1!p_2!\cdots p_N!}\right)^2, \tag{1}$$

$N \geq 2$, $\boldsymbol{\alpha} = \{\alpha_1,\ldots,\alpha_N\} \in \mathbb{C}^N$ is fixed, and in the sum p_i are always non-negative integers. We will use the notation $\binom{n}{p_1,\ldots,p_N} = \left(\frac{n!}{p_1!p_2!\cdots p_N!}\right)$, write $a_n^{\alpha_1,\ldots,\alpha_N} = a_n^{\{\alpha_1,\ldots,\alpha_N\}}$, and use the notation $a_n^N := a_n^{1,\ldots,1}$, where the superscript is a list of N ones. This should cause no confusion, since we do not allow $N=1$ in definition (1), as that case is too easy.

The simplest non-trivial example is the sequence $a_n^{1,1}$, for which we have the following sequence (#A000984 in Sloan's tables [S]).

$$a_n^2 := a_n^{1,1} := \sum_{p=0}^{n} \binom{n}{p}^2 = \binom{2n}{n}. \tag{2}$$

This is a well known result, a special case of $\sum_{p+q=C} \binom{A}{p}\binom{B}{q} = \binom{A+B}{C}$, the number of ways of taking C elements from a set which is the union of two sets of sizes A and B, by summing over the number of elements taken from each set.

Another example, similar in appearance, is the case of $a_n^{1,-1}$, where

$$a_n^{1,-1} = \sum_{p=0}^{n} (-1)^{n-p} \binom{2n}{p}^2 = \binom{2n}{n}. \tag{3}$$

Examples (2) and (3) may be found in many texts on combinatorics, and also in the survey [W]. Along these lines we also have the less obvious result

$$a_n^{1,\omega,\omega^2} := \sum_{p+q+r=3m} \omega^{p-q} \binom{3m}{p,q,r}^2 = \binom{4m}{2m,m,m}, \tag{4}$$

where ω is a primative third root of unity. Proofs of (3) and (4) (given in Examples 12 and 16 respectively) will follow from the general recurrence relations we find, for example, Theorem 8, which says that in the $N = 2$ case we have

$$na_n^{a,b} - (2n-1)(a+b)a_{n-1}^{a,b} + (n-1)(a-b)^2 a_{n-2}^{a,b} = 0. \tag{5}$$

Theorem 13 gives a similar formula for the case $N = 3$. Corollary 6 states that we can always obtain recurrence relations, with at most $2^{N-1}+1$ terms, and Section 3 gives a concrete method of finding these, which does not make use of computing any terms, and thus is faster than methods which find recurrences given only the existance of a recurrence of known degree, and the terms of the sequence. Theorem 1 gives an explicit formula for the recurrence for all N, in the case that all α_i are 1, and a few examples obtained by plugging in given values of N are given in Table 1.

Geometrical motivation

The motivation for studying the expressions given in the title of this note comes from the study of the family of Calabi-Yau varieties, the general member of which is given by the resolution of a hypersurface in $\mathbb{P}^{N-1}$ defined by the following equation—which actually should be multiplied through on both sides by $X_1 \ldots X_N$ to obtain a homogeneous equation of degree $N-1$.

$$\mathcal{X}_t^{\alpha_1,\ldots,\alpha_N} : \quad (X_1 + \cdots + X_N)\left(\frac{\alpha_1}{X_1} + \cdots + \frac{\alpha_N}{X_N}\right) t = 1. \tag{6}$$

Here t is the parameter of the family, and we take α_i to be nonzero fixed elements of $\mathbb{C}$. Aspects of the $N = 3$ case are considered in [V1] and [V2], and the $N = 4$ case is studied in [HV]. Some work on the general case can be found in [Ludwig].

In order to study a family of varieties, one often works with the periods of the members. Using the methods of [PS], in the $N = 3, \alpha_i = 1$ case, it was shown in [V1] that one of the periods of this family has the form *constant* $\times \sum_{n\geq 0} a_n^{1,1,1} t^n$. Exactly the same considerations show that in general, up to a constant factor, $\sum_{n\geq 0} a_n^{\alpha_1,\dots,\alpha_N} t^n$ is a period for $\mathcal{X}_t^{\alpha_1,\dots,\alpha_N}$. (See abstract or below for definitions of the a_n.)

The Picard-Fuchs equation is a differential equation satisfied by the periods. One method to find this equation (used for example in [PS]) is to find a recurrence relation for the coefficients a_n, which is what the rest of this note is devoted to.

The rest of this paper is more combinatorial than geometric. We return to the geometry in Section 5, where we give the Picard-Fuchs equation for the families of elliptic curves, corresponding to the $N = 3$ case (43). However, we do not discuss the geometry or modularity of the varieties $\mathcal{X}_t^{\alpha_1,\dots,\alpha_N}$ for $N > 3$ here, which is currently work in progress. We do give differential equations (16) closely related to the Picard-Fuchs equations in the case where all $\alpha_i = 1$; Table 2 gives some examples.

2 The case $\alpha_1 = \cdots = \alpha_N = 1$

We first consider the case $\alpha_1 = \cdots \alpha_N = 1$, since this is simpler, and in this case we can obtain an explicit formula for the recurrence relation. In order to find a recurrence relation for

$$a_n^N := \sum_{\sum p_i = n} \left(\frac{n!}{p_1! p_2! \cdots p_N!} \right)^2,$$

we introduce auxiliary sequences of numbers $a_n^{N,j}$, for $0 \leq j \leq N$, defined by

$$a_n^{N,j} := \sum_{\sum p_i = n} \left(\frac{n!}{p_1! p_2! \cdots p_N!} \right)^2 p_1 \cdots p_j. \tag{7}$$

We have $a_n^N = a_n^{N,0}$, and for convenience, we set $a_n^{N,N+1} = a_n^{N,-1} = 0$.

It is easy to verify that for $0 \leq j \leq N$,

$$n a_n^{N,j} = (N - j) a_n^{j+1} + j n^2 a_{n-1}^{N,j-1}. \tag{8}$$

Rearranging this equation, we have

$$(N - j) a_n^{N,j+1} = n a_n^{N,j} - j n^2 a_{n-1}^{N,j-1}. \tag{9}$$

$$
\begin{aligned}
0 &= na_n^2 - 2(2n-1)a_{n-1}^2 \\
0 &= n^2a_n^3 - (10n^2-10n+3)a_{n-1}^3 + 9(n-1)^2a_{n-2}^3 \\
0 &= n^3a_n^4 - 2(2n-1)(5n^2-5n+2)a_{n-1}^4 + 64(n-1)^3a_{n-2}^4 \\
0 &= n^4a_n^5 - (35n^4-70n^3+63n^2-28n+5)a_{n-1}^5 \\
&\quad +(n-1)^2(259(n-1)^2+26)a_{n-2}^5 - (3\cdot 5)^2(n-1)^2(n-2)^2a_{n-3}^5 \\
0 &= n^5a_n^6 - 2(2n-1)(14n^4-28n^3+28n^2-14n+3)a_{n-1}^6 \\
&\quad +4(n-1)^3(196(n-1)^2+59)a_{n-2}^6 \\
&\quad -(2\cdot 4\cdot 6)^2(n-1)^2(n-2)^2(n-\tfrac{1}{2})^2a_{n-3}^6 \\
0 &= n^6a_n^7 - (84(n(n-1))^3+126(n(n-1))^2+54n(n-1)+7)a_{n-1}^7 \\
&\quad +3(n-1)^2(658(n-1)^4+396(n-1)^2+17)a_{n-2}^7 \\
&\quad -2(n-1)^2(n-2)^2(6458n^2-19374n+15505)a_{n-3}^7 \\
&\quad +(3\cdot 5\cdot 7)^2(n-1)^2(n-2)^2(n-3)^2a_{n-4}^7 \\
0 &= n^7a_n^8 \\
&\quad -2(2n-1)(30(n(n-1))^3+54(n(n-1))^2+27n(n-1)+4)a_{n-1}^8 \\
&\quad +12(n-1)^3(364(n-1)^4+365(n-1)^2+47)a_{n-2}^8 \\
&\quad -2^7(n-1)^2(n-2)^2(2n-3)(205n^2-615n+554)a_{n-3}^8 \\
&\quad +(2\cdot 4\cdot 6\cdot 8)^2(n-1)^2(n-2)^3(n-3)^2a_{n-4}^8 \\
0 &= n^8a_n^9 \\
&\quad -(165n^8-660n^7+1386n^6-1848n^5+1650n^4-990n^3+385n^2-88n+9)a_{n-1}^9 \\
&\quad +3(n-1)^2(2926n^6-17556n^5+48290n^4-76120n^3+71423n^2-37422n+8487)a_{n-2}^9 \\
&\quad -(n-2)^2(n-1)^2(172810n^4-1036860n^3+2489234n^2-2801832n+1237167)a_{n-3}^9 \\
&\quad +9(n-3)^2(n-2)^2(n-1)^2(117469n^2-469876n+493542)a_{n-4}^9 \\
&\quad -(3\cdot 5\cdot 7\cdot 9)^2(n-4)^2(n-3)^2(n-2)^2(n-1)^2a_{n-5}^9 \\
0 &= n^9a_n^{10} \\
&\quad -2(2n-1)(55n^8-220n^7+484n^6-682n^5+649n^4-418n^3+176n^2-44n+5)a_{n-1}^{10} \\
&\quad 4(n-1)^3(4092n^6-24552n^5+69993n^4-116292n^3+116754n^2-66396n+16675)a_{n-2}^{10} \\
&\quad -8(n-2)^2(n-1)^2(2n-3)(30580n^4-183480n^3+458909n^2-551067n+267900)a_{n-3}^{10} \\
&\quad 256(n-3)^2(n-2)^3(n-1)^2(21076n^2-84304n+97035)a_{n-4}^{10} \\
&\quad -(2\cdot 4\cdot 6\cdot 8\cdot 10)^2(n-4)^2(n-3)^2(n-2)^2(n-1)^2(n-\tfrac{5}{2})a_{n-5}^{10}
\end{aligned}
$$

Table 1: Recurrence relations for a_n^N, from Equation (12) for $2 \leq N \leq 10$

$$\begin{tikzcd}
g(N,4)a_n^{N,5} \arrow[r, "n"] \arrow[dr, "-4(N-3)n^2"] & g(N,3)a_n^{N,4} \arrow[r, "n"] \arrow[dr, "-3(N-2)n^2"] & g(N,2)a_n^{N,3} \arrow[r, "n"] \arrow[dr, "-2(N-1)n^2"] & g(N,1)a_n^{N,2} \arrow[r, "n"] \arrow[dr, "-(N)n^2"] & g(N)a_n^{N,1} \arrow[r, "n"] & a_n^N \\
& g(N,2)a_{n-1}^{N,3} \arrow[r, "n-1"] \arrow[dr, "-2(N-1)(n-1)^2"] & g(N,1)a_{n-1}^{N,2} \arrow[r, "n-1"] \arrow[dr, "-(N)(n-1)^2"] & g(N)a_{n-1}^{N,1} \arrow[r, "n-1"] & a_{n-1}^N & \\
& & Na_{n-2}^{N,1} \arrow[r, "n-2"] & a_{n-2}^N & &
\end{tikzcd}$$

Figure 1: Diagram of relations (9) between $a_n^{N,j}$ (7)

Now, starting with $j = N$, and applying this relation repeatedly, we will obtain a recurrence relation for the a_n. We can visualise this process in Figure 1, drawn up to $a_n^{N,5}$, where we use the notation

$$g(N,j) = (N-1)(N-2)\cdots(N-j).$$

In Figure 1, the configuration

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[dr, "b"'] & A \\
& B
\end{tikzcd} \tag{10}$$

means $X = aA + bB$. Thus to find an expression for $g(N,4)a_n^{N,5}$ in terms of the a_n^N, we just have to follow all paths from $g(N,4)a_n^{N,5}$ to the $a_n^N, a_{n-1}^N, a_{n-2}^N$, multiplying the coefficients on the arrows in any path, and summing the products for paths ending at the same point. In this example, we obtain

$$\begin{aligned}
g(N,4) \quad = \quad & n^5 a_n^N \\
& -\Big(Nn^5 + 2(N-1)n^4(n-1) \\
& \qquad\qquad +3(N-2)n^3(n-1)^2 + 4(N-3)n^2(n-1)^3\Big)a_{n-1}^N \\
& +\Big(3(N-2)Nn(n-1)^2 + 4N(N-3)(n-1)^3 \\
& \qquad\qquad +8(N-3)(N-1)(n-1)^2(n-2)\Big)a_{n-2}^N.
\end{aligned}$$

This is true for all $N \geq 1$. However, for $N = 4$, we have $g(4,4) = 0$, and so, substituting $N = 4$, and dividing by n^2, we obtain the relation:

$$n^3 a_n^4 - 2(n-1)(5n^2 - 5n + 2)a_{n-1}^4 + 64(n-3)^3 a_{n-2}^4 = 0.$$

The first few cases of $g(N,j)a_n^{N,j+1}$ are as follows.

$$\begin{aligned}
g(N,0)a_n^{N,1} &= na_n^N \\
g(N,1)a_n^{N,2} &= n^2a_n^N - Nn^2a_{n-1}^N \\
g(N,2)a_n^{N,3} &= n^3a_n^N - \big[(3N-2)n-(2N-2)\big]n^2a_{n-1}^N \\
g(N,3)a_n^{N,4} &= n^4a_n^N - [(6N-8)n^2+(-8N+14)n+3(N-2)]n^2a_{n-1}^N \\
&\quad +3N(N-2)(n-1)^2n^2a_{n-2}^N
\end{aligned}$$

In general, the formula we obtain is:

$$g(N,j)a_n^{N,j+1} = \sum_{k\geq 0}\left[n^{j+1} \sum_{\substack{1\leq i\leq k \\ \alpha_i+\beta_i=N+1,\alpha_i\in\mathbb{N} \\ 1<\alpha_{i+1}+1<\alpha_i\leq j}} \prod_{i=1}^{k} -\alpha_i\beta_i\left(\frac{n-i}{n-i+1}\right)^{\alpha_i-1}\right] a_{n-k}^N, \tag{11}$$

where an empty product is taken to be 1. Note that this is a finite sum, since for $k > \lfloor N/2 \rfloor$, there are no possible sequences of $\alpha_1,\ldots,\alpha_k$ with $1\leq\alpha_k<\alpha_1\leq j$ and $\alpha_{i+1}\leq\alpha_i-2$. Note also that since $\alpha_i>\alpha_{i+1}$ this is in fact a polynomial in n.

Since $g(N,N)=0$, substituting $j=N$ in (11), we obtain the following result.

Theorem 1. *For any positive integer N, the sequence $a_n^N=\sum\left(\frac{n!}{p_1!p_2!\cdots p_N!}\right)^2$ (where the sum is over non-negative integers $p_1,\ldots,p_N$ summing to n) satisfies the recurrence relation*

$$\sum_{k\geq 0}\left[n^{N+1} \sum_{\substack{1\leq i\leq k \\ \alpha_i+\beta_i=N+1,\alpha_i\in\mathbb{N} \\ 1<\alpha_{i+1}+1<\alpha_i\leq N}} \prod_{i=1}^{k} -\alpha_i\beta_i\left(\frac{n-i}{n-i+1}\right)^{\alpha_i-1}\right] a_{n-k}^N = 0. \tag{12}$$

The recurrences in the first few cases, (after dividing by n^2), are given in Table 1.

Remark 2. The coefficients of a_{n-k}^N in (12) may be written in other forms, for example, the coefficient of a_{n-1}^N can also be written as:

$$N\left(n^{N+2}-(n-1)^{N+2}\right)-(N+2)n(n-1)\left(n^N-(n-1)^N\right). \tag{13}$$

Remark 3. Formula (12) is a closed formula. However, alternatively, using (9), one can quickly produce generating functions for the recurrences as

follows. Define a sequence of polynomials in n, N, x by $p_{-1}(N,n,x) = 1$, $p_0(N,n,x) = n$, and

$$p_j(N,n,x) := np_{j-1}(N,n,x) - jn^2(N-j+1)xp_{j-2}(N,n-1,x). \quad (14)$$

Then $p_j(N,n,x)$ is a polynomial with coefficients given by the coefficients of a^N_{n-k} in (11). The terms of the recurrence for a^N_n are given by coefficients of x in $p_N(N,n,x)$, so we have

$$\sum_{k=0}^{\lfloor\frac{N+1}{2}\rfloor} c^N_k(n)a^N_{n-k} = 0, \text{ where } p_N(N,n,x) = \sum c^N_r(n)x^r. \quad (15)$$

From (12), using the standard method we obtain the following differential equation for $f_N(x) = \sum_{n\geq 0} a^N_n x^n$, where $\Theta = x\frac{d}{dx}$.

$$\mathcal{F}_N = \sum_{k\geq 0} t^k \sum_{\substack{1\leq i\leq k,\, \alpha_i+\beta_i = N+1,\\ \alpha_i\in\mathbb{N},\, \alpha_{k+1}=0,\\ 1<\alpha_{i+1}+1<\alpha_i\leq N,}} (\Theta+k)^{N+1-\alpha_1}\prod_{i=1}^{k} -\alpha_i\beta_i\left(\Theta+k-i\right)^{\alpha_i-\alpha_{i+1}}. \quad (16)$$

The first few cases (up to sign) are given in Table 2.

Equations $\mathcal{F}_3$ is the Picard-Fuchs equation of the family of elliptic curves for $\Gamma_1(6)$, and appears in [SB, Table 7]. $\mathcal{F}_4$ is the Picard-Fuchs equation of the A_3 family of K3 surfaces studied in [V1], where $\mathcal{F}_5$ is also given. $\mathcal{F}_5$ and $\mathcal{F}_6$ can be found in examples #34 and #130 respectively in the tables of [AZ].

3 Weighted sums

Now we want to find a recurrence relation for the terms

$$a_n^{\alpha_1,\ldots,\alpha_N} := \sum_{\sum p_i = n}\left(\prod_{i=1}^{N}\alpha_i^{p_i}\right)\left(\frac{n!}{p_1!p_2!\cdots p_N!}\right)^2. \quad (17)$$

We will call these weighted sums, as opposed to the case where all $\alpha_i = 1$.

Finding a recurrence relation for these a_n is achieved in a similar manner as for the previous case, but now the situation is slightly more complicated. In particular, the version of (8) cannot in this case be so easily reversed to give a relation like (9) (this can be done for $N = 2$, but for large N the situation becomes too complicated). So, although we will obtain a diagram (Figure 2) similar to Figure 1 our arrows will now be in the opposite direction. But we will still be able to obtain a relation using the finite dimensionality of certain vector spaces.

$$
\begin{aligned}
\mathcal{F}_2 &= (4x-1)\Theta+2x \\
\mathcal{F}_3 &= (9x-1)(x-1)\Theta^2+2x(9x-5)\Theta+3x(3x-1) \\
\mathcal{F}_4 &= (16x-1)(4x-1)\Theta^3+6x(32x-5)\Theta^2+6x(32x^2-3)\Theta+4x(16x-1) \\
\mathcal{F}_5 &= (25x-1)(9x-1)(x-1)\Theta^4+x(1350x^2-1036x+70)\Theta^3 \\
&\quad +x(2925x^2-1580x+63)\Theta^2+x(2700x^2-1088x+28)\Theta \\
&\quad +5x(180x^2-57x+1) \\
\mathcal{F}_6 &= (4x-1)(16x-1)(36x-1)\Theta^5+(17280x^3-3920x^2+140x)\Theta^4 \\
&\quad +(50688x^3-8076x^2+168x)\Theta^3+(72576x^3-8548x^2+112x)\Theta^2 \\
&\quad +(50688x^3-4628x^2+40x)\Theta+6x(2304x^2-170x+1) \\
\mathcal{F}_7 &= (49x-1)(25x-1)(9x-1)(x-1)\Theta^6 \\
&\quad +(132300x^4-116244x^3+11844x^2-252x)\Theta^5 \\
&\quad +(639450x^4-431406x^3+30798x^2-378x)\Theta^4 \\
&\quad +(1587600x^4-844776x^3+44232x^2-336x)\Theta^3 \\
&\quad +(2127825x^4-919770x^3+36789x^2-180x)\Theta^2 \\
&\quad +(1455300x^4-527112x^3+16698x^2-54x)\Theta \\
&\quad +7x(56700x^3-17720x^2+459x-1)
\end{aligned}
$$

Table 2: Differential equations for $\sum_{n\geq 0} a_n^N t^n$ for $2 \leq N \leq 7$, obtained as special cases of (16)

Our auxiliary terms in this case are defined by

$$a_n^{\varepsilon_1,\ldots,\varepsilon_N} := \sum_{\sum p_i = n} p_1^{\varepsilon_1} \cdots p_N^{\varepsilon_N} \left(\prod_{i=1}^{N} \alpha_i^{p_i} \right) \left(\frac{n!}{p_1! p_2! \cdots p_N!} \right)^2, \tag{18}$$

where $\varepsilon_k \in \{1, -1\}$, which reduces to (7) in the case that all $\alpha_i = 1$, with j in (7) given by $j = \sum \varepsilon_k$.

The generalisation of (8) is given by

$$\begin{aligned} n a_n^{\varepsilon_1,\ldots,\varepsilon_N} &= \sum_{i=1}^{N} (1 - \varepsilon_i) a_n^{\varepsilon_1,\ldots,\varepsilon_i+1,\ldots,\varepsilon_N} \\ &\quad + n^2 \sum_{i=1}^{N} \alpha_i \varepsilon_i a_{n-1}^{\varepsilon_1,\ldots,\varepsilon_i-1,\ldots,\varepsilon_N} \end{aligned} \tag{19}$$

Notice that since $\varepsilon_i \in \{-1, 1\}$, the right hand side of this expression always has N terms.

The "direction" of this equation can not easily be reversed. Instead of a diagram as in Figure 1, built of figures of the form (10), we would have building blocks of a much more complicated form, e.g., such as:

$$\begin{tikzcd} a_n^{1,1,0} & \\ & a_n^{1,0,0} \arrow[ul, "1/n"'] \arrow[dl, "1/n"] \arrow[d, "n\alpha_1"] \\ a_n^{1,0,1} & a_{n-1}^{0,0,0} \end{tikzcd} \tag{20}$$

meaning

$$a_n^{1,0,0} = \tfrac{1}{n} a_n^{1,0,1} + \tfrac{1}{n} a_n^{1,1,0} + n\alpha_1 a_{n-1}^{0,0,0}.$$

In order to work with these relations, we consider N fixed, and define vector spaces

$$V_n^j = \bigoplus_{\sum \varepsilon_i = j} K a_n^{\varepsilon_1,\ldots,\varepsilon_N}, \tag{21}$$

where K is the field of fractions of $\mathbb{Z}[n, \alpha_1, \ldots, \alpha_n]$, and n is considered an independent variable, and the a_n satisfy only the relations given by (19). We now consider (19) as defining a map $V_n^j \to V_n^{j+1} \oplus V_{n-1}^{j-1}$. Summing such expressions together, we define maps

$$\Phi_n^j : W_n^j \to W_{n-j}^{1-j}, \text{ where } j = 0 \text{ or } 1, \text{ and where } W_n^j := \bigoplus_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} V_{n-k}^{2k+j}. \tag{22}$$

We define the composition

$$\Psi_n = \Phi_n^1 \circ \Phi_n^0. \tag{23}$$

$$
\begin{tikzcd}
 & & V_n^4 \arrow[d, dotted] \arrow[dr, dash, "\oplus" description] & V_n^3 \arrow[l, dotted] \arrow[d, dotted] & V_n^2 \arrow[l, dotted] \arrow[d, dotted] & V_n^1 \arrow[l, dotted] \arrow[d, dotted] & V_n^0 \arrow[l, dotted] \\
 & V_{n-1}^4 \arrow[d, dotted] \arrow[dr, dash, "\oplus" description] & V_{n-1}^3 \arrow[l, dotted] \arrow[d, dotted] \arrow[dr, dash, "\oplus" description] & V_{n-1}^2 \arrow[l, dotted] \arrow[d, dotted] \arrow[dr, dash, "\oplus" description] & V_{n-1}^1 \arrow[l, dotted] \arrow[d, dotted] & V_{n-1}^0 \arrow[l, dotted] & \\
V_{n-2}^4 \arrow[d, dotted] & V_{n-2}^3 \arrow[l, dotted] \arrow[d, dotted] & V_{n-2}^2 \arrow[l, dotted] \arrow[d, dotted] \arrow[dr, dash, "\oplus" description] & V_{n-2}^1 \arrow[l, dotted] \arrow[d, dotted] \arrow[dr, equal] & V_{n-2}^0 \arrow[l, dotted] \arrow[dr, equal] & & \\
{} & {} & {} & V_{n-3}^0 \arrow[l, dotted] \arrow[dr, equal] & W_{n-2}^1 \arrow[d, "\Phi_{n-2}^1"] & W_{n-2}^0 \arrow[l, "\Phi_{n-2}^0"'] \arrow[dl, bend left, "\Psi_{n-2}"] & \\
 & & & & W_{n-3}^0 & &
\end{tikzcd}
$$

Figure 2: Maps used to find the $a_N^{\alpha_1,\ldots,\alpha_N}$ recurrence relation

We can visualise this, for example in the case of $N = 4$ in the diagram in Figure 2, which shows only part of an infinite diagram. This diagram corresponds to Figure 1, with $g(N,4)a_n^{N,5}$ removed, since we will set $j = N$ so that this is zero, and with the $a_n^{N,j}$ replaced by V_n^j, since in the special case where all $\alpha_i = 1$, the V_n^j would be one dimensional, spanned by $a_n^{N,j}$.

Lemma 4. $\dim W_n^j \leq 2^{N-1}$.

Proof. From the definition (21), $\dim V_n^j \leq \binom{N}{j}$ (with equality only if the a_n^ε spanning V_n are independent over K), and so from the definition (22), the dimension of W_n^j is at most the sum of alternative binomial coefficients. These sums represent the number of ways of taking an even, respectively odd, subset from a set of N items. Subsets correspond to binary sequences in $(\mathbb{Z}/2\mathbb{Z})^N$, so there is a bijection between these sets corresponding to the map $\mathbf{x} \mapsto \mathbf{x} + (1, 0, \ldots, 0)$. $\square$

Corollary 5. *For $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$, if $a_n = a_n^{\alpha_1,\ldots,\alpha_N}$ as defined in (17), then $a_n, a_{n-1}, \ldots, a_{n-2^{N-1}}$ satisfy a linear relation over K.*

Proof. This is because the maps Φ_n^j (22) are injective, since they are defined by equalities (19). A linear relationship (which exists by Lemma 4) between $\Psi_{n-2^{N-1}+1} \circ \cdots \circ \Psi_n(a_n), \Psi_{n-2^{N-1}+1} \circ \cdots \circ \Psi_{n-1}(a_{n-1}), \ldots, a_{n-2^{N-1}}$ in $W_{n-2^{N-1}}$ thus pulls back to a relationship between the a_n. $\square$

Viewing a_n as numbers rather than independent variables, one obtains immediately the following result.

Corollary 6. *There is a recurrence relation satisfied by the* $a_n^{\alpha_1,\ldots,\alpha_N}$*, with at most* $2^{N-1}+1$ *terms, and with coefficients given as polynomials in* n *and* $\alpha_1,\ldots,\alpha_N$*.*

Remark 7. By making a more careful analysis of the entries of the matrices Ψ_i, it should be possible to obtain a bound on the degree of the coefficients (as polynomials in n) of the recurrence relation.

Note that the recurrence relation may have fewer than $2^{N-1}+1$ terms—by Lemma 4, this is determined by the dimension of the vector spaces W_n^j, which depends on the relationships between the α_i.

4 Formulae and examples

Although the above results give a procedure for determining a recurrence relation, we still do not have a general formula comparable with (12). The recurrence relations must be determined case by case. For a given set of α_i, since we now know a bound on the degree and number of terms of a recurrence relation, it is a simple matter of linear algebra to determine the relation explicitly. However, what makes this result more interesting is that we can obtain formulas for an infinite number of cases at once.

4.1 The recurrence when $N=2$

In the case $N=2$, applying the above method we have the following result.

Theorem 8. *For* $a,b\in\mathbb{C}$*, the terms*

$$a_n^{a,b} := \sum_{p+q=n} a^p b^q \binom{n}{p}^2$$

satisfy a recurrence relation

$$n a_n^{a,b} - (2n-1)(a+b)a_{n-1}^{a,b} + (n-1)(a-b)^2 a_{n-2}^{a,b} = 0. \qquad (24)$$

Remark 9. Note that substituting $a=b=1$ into (24) gives the same result as substituting $N=2$ into (12) in Theorem 1.

Proof. In this case from (19) we have

$$\begin{aligned} n a_n^{1,1} &= n^2 a a_{n-1}^{0,1} + n^2 b a_{n-1}^{1,0} \\ n a_n^{0,0} &= a_n^{1,0} + a_n^{0,1} \\ n a_n^{1,0} &= a_n^{1,1} + n^2 a a_{n-1}^{0,0} \\ n a_n^{0,1} &= a_n^{1,1} + n^2 b a_{n-1}^{0,0}. \end{aligned}$$

In matrix notation, these are:

$$\begin{pmatrix} a_n^{0,0} \\ a_{n+1}^{1,1} \end{pmatrix} = \begin{pmatrix} 1/n & 1/n \\ (n+1)b & (n+1)a \end{pmatrix} \begin{pmatrix} a_n^{1,0} \\ a_n^{0,1} \end{pmatrix} \tag{25}$$

$$\begin{pmatrix} a_n^{1,0} \\ a_n^{0,1} \end{pmatrix} = \begin{pmatrix} na & 1/n \\ nb & 1/n \end{pmatrix} \begin{pmatrix} a_{n-1}^{0,0} \\ a_n^{1,1} \end{pmatrix} \tag{26}$$

Thus the composition (23) is given by

$$\Psi_n = \begin{pmatrix} na & nb \\ 1/n & 1/n \end{pmatrix} \begin{pmatrix} 1/n & (n+1)b \\ 1/n & (n+1)a \end{pmatrix} = \begin{pmatrix} a+b & 2abn(n+1) \\ 2/n^2 & (a+b)(n+1)/n \end{pmatrix},$$

and (writing $*$ for entries we are not interested in)

$$\Psi_{n-1} = \begin{pmatrix} a+b & * \\ 2/(n-1)^2 & * \end{pmatrix}, \Psi_{n-1} \circ \Psi_n = \begin{pmatrix} (a+b)^2 + 4ab(n-1)/n & * \\ \frac{2(2n-1)(a+b)}{(n-1)^2 n} & * \end{pmatrix}.$$

Now we can see that the linear relationship referred to in Corollary 5, which now is a relationship in W_{n-2} spanned by $a_{n-2}^{0,0}$ and $a_{n-1}^{1,1}$, is given by

$$n \begin{pmatrix} (a+b)^2 + \frac{4ab(n-1)}{n} \\ \frac{2(2n-1)(a+b)}{(n-1)^2 n} \end{pmatrix} - (2n-1)(a+b) \begin{pmatrix} a+b \\ \frac{2}{(n-1)^2} \end{pmatrix} + (n-1)(a-b)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.$$

The recurrence relation follows, as described in Corollary 5. $\square$

Corollary 10. *The function $f = \sum_{n \geq 0} a_n^{a,b} t^n$ satisfies the following differential equation:*

$$\left[(1 - 2(a+b)t + t^2(a-b)^2)\Theta - t(a+b) + t^2(a-b)^2\right] f = 0.$$

Remark 11. Notice that the coefficient of Θ factors as $((\sqrt{a} + \sqrt{b})^2 t - 1)((\sqrt{a} - \sqrt{b})^2 t - 1)$. This is an example of a general phenomena, which has a geometric explanation, given by the determination of the singular members of the family $\mathcal{X}_t^{\alpha_1, \ldots, \alpha_N}$; the A_4 situation is given in [HV, Lemma 3.7], and the general case works in exactly the same way.

Example 12. For $a = b = 1$ in (24) in Theorem 8 we obtain the recurrence

$$n a_n^{1,1} = 2(2n-1) a_{n-1}^{1,1}. \tag{27}$$

In the introduction we remarked that $a_n^{1,1} = \binom{2n}{n}$. In the case $a = -b = 1$ we have

$$n a_n^{1,-1} = 4(n-1) a_{n-2}^{1,-1}. \tag{28}$$

From the definition (1) we see that $a_1^{1,-1} = 0$, so from (28) all odd terms are zero. Setting $c_n = a_{2n}^{1,-1}$ and substituting in (28) we obtain $2na_2n^{1,-1} = 4(2n-1)a_{2n-2}^{1,-1}$, and then

$$nc_n = 2(2n-1)c_n \tag{29}$$

Since this is the same relation as (12), and $a_0^{1,1} = c_0 = 1$, we have $a_n = c_n$, giving us the formula (3) given in the introduction. Note, both these examples are well known. Refer also to [C] and [].

4.2 The recurrence when $N = 3$

In the case of $N = 3$, the matrices we are interested in are given by

$$\Phi_n^0 = \begin{pmatrix} 0 & 1/n & 1/n & 1/n \\ 1/(n+1) & 0 & (n+1)c & (n+1)b \\ 1/(n+1) & c(n+1) & 0 & (n+1)a \\ 1/(n+1) & b(n+1) & (n+1)a & 0 \end{pmatrix}^t$$

and

$$\Phi_n^1 = \begin{pmatrix} 0 & (n+1)a & (n+1)b & (n+1)c \\ na & 0 & 1/n & 1/n \\ nb & 1/n & 0 & 1/n \\ nc & 1/n & 1/n & 0 \end{pmatrix}^t.$$

From this, setting $A = a+b+c$, we obtain

$$\Psi_n = \begin{pmatrix} A & 2cbn(n+1) & 2acn(n+1) & 2ban(n+1) \\ 2/n^2 & A+(b+c)/n & 2a+a/n & 2a+a/n \\ 2/n^2 & 2b+b/n & A+(a+c)/n & 2b+b/n \\ 2/n^2 & 2c+c/n & 2c+c/n & A+(a+b)/n \end{pmatrix}^t.$$

In W_{n-4}, $\Psi_{n-3} \circ \cdots \circ \Psi_n(a_n), \ldots, a_{n-4}$ are given by the columns of the following matrix (computer algebra packages e.g., [Pari], [Magma], are helpful in obtaining and manipulating this matrix).

$$\begin{bmatrix} H_0 & H_1 & A^2 + \frac{4B(n-3)}{n-2} & A & 1 \\ G_2(a,b,c) & G_1(a,b,c) & G_0(a,b,c,n-2) & \frac{2}{(n-3)^2} & 0 \\ G_2(b,c,a) & G_1(b,c,a) & G_0(b,c,a,n-2) & \frac{2}{(n-3)^2} & 0 \\ G_2(c,a,b) & G_1(c,a,b) & G_0(c,a,b,n-2) & \frac{2}{(n-3)^2} & 0 \end{bmatrix} \tag{30}$$

where $A = a+b+c$, $B = ab+bc+ca$, and

$$\begin{aligned} H_0 &= A^4 + \frac{4A^2B(6n^3-28n^2+37n-12)}{n(n-1)(n-2)} + \frac{16B^2(n-1)(n-3)}{n(n-2)} \\ &\quad + \frac{4Aabc(10n^2-10n+3)(4n-7)(n-3)}{n^2(n-1)^2} + \frac{12Aabc(4n-3)(2n-5)}{n^2}, \\ H_1 &= A^3 + \frac{4(3n^2-13n+13)AB}{(n-1)(n-2)} + \frac{12(n-3)(4n-7)abc}{(n-1)^2}, \end{aligned}$$

$$\begin{aligned}
G_0(a,b,c,n) &= \tfrac{2(n-1)(4n-3)a+2An(2n-1)}{n^2(n-1)^2},\\
G_1(a,b,c) &= \tfrac{(4n-7)(10A-4a)a}{(n-1)^2(n-3)} - \tfrac{6(2n-3)Aa}{(n-1)^2(n-2)^2(n-3)} - \tfrac{2(n-2)(A^2-4B)}{(n-1)(n-3)^2} + \tfrac{2(2n-5)(2n-3)A^2}{(n-1)(n-2)(n-3)^2},\\
G_2(a,b,c) &= \Big[A^2 + \tfrac{4B(n-1)}{n}\Big]\tfrac{(2(n-3)(4n-11)a+2A(n-2)(2n-5))}{(n-2)^2(n-3)^2}\\
&-A(2n-1)\tfrac{(n-3)4(6n^2-22n+19)bc-2(n-3)(12n^2-50n+47)a^2-2B(2n-5)^2(5n-7)}{n(n-1)^2(n-3)^2(n-2)}\\
&+a(4n-3)\tfrac{4cb(12n^3-71n^2+138n-87)-2(n-3)(2n-3)a^2+8(a^2+b^2+c^2)(n-2)^2(n-3)+2B(n-3)(2n-5)(4n-5)}{(n-3)^2(n-2)(n-1)n^2}\\
&-2\Big[(4n-3)(n-1)a+An(2n-1)\Big]\tfrac{(2n-3)(2n-5)(a^2+cb)+(b+c)a(4n-7)+A^2(-n^2+3n-2)}{(n-3)(n-2)(n-1)^2n^2}.
\end{aligned}$$

In general (30) has rank 4, which can be verified by plugging in any values for a, b, c. If exactly two of a, b, c are equal, then two of the last three rows are equal, so the rank is 3; if $a = b = c$ the rank is 2. By the same method as for Theorem 8 a recurrence relation is obtained by finding linear relations between the columns, i.e., by finding the kernel of this matrix. Generally this gives a 5 term recurrence, but if $a = b$, or $a = b = c$, there will be a relation between the last 4 or 3 columns respectively. By explicitly computing the kernel (again, aided by [Magma] and [Pari]) we obtain the following result.

Theorem 13. *For $a, b, c \in \mathbb{C}$, $abc \neq 0$ the terms*

$$a_n = a_n^{a,b,c} := \sum_{p+q+r=n} a^p b^q c^r \binom{n}{p, q, r}^2 \tag{31}$$

satisfy a recurrence relation $R(n) = 0$ where

$$\begin{aligned}
R(n) := F_{11}F_7 n^2 a_n - AF_{11}\Big[2(n-1)(2n-3)(4n-1)+3\Big]a_{n-1}\\
+\Big[(2n-3)^2F_{11}F_3A^2 + [F_3F_{11}(n-2)(n-1)-3]\,s\Big]a_{n-2}\\
-F_9F_3\Big[sA(4n^2-18n+19)+4abcF_{11}F_7\Big]a_{n-3} + F_7F_3s^2(n-3)^2a_{n-4},
\end{aligned} \tag{32}$$

and $F_p = 4n - p$, $A = a + b + c$ and $s = a^2 + b^2 + c^2 - 2ab - 2ac - 2cb$.

Remark 14. The differential equation for $\sum a_n t^n$ corresponding to (32) has degree 4 in $\Theta = \frac{td}{dt}$; the coefficient of Θ^4 is $\prod((\sqrt{a} \pm \sqrt{b} \pm \sqrt{c})^2 t - 1)$.

Example 15. If $s = a^2 + b^2 + c^2 - 2ab - 2ac - 2cb = 0$, which happens when $\sqrt{a} \pm \sqrt{b} \pm \sqrt{c} = 0$, then (32) is divisible by F_{11}. Since n is an integer, so F_{11} is never 0, we obtain

$$\begin{aligned}
0 = F_7n^2a_n - A\,[2(n-1)(2n-3)(4n-1)+3]\,a_{n-1}\\
+A^2(2n-3)^2F_3a_{n-2} - 4abcF_9F_3F_7a_{n-3}.
\end{aligned}$$

Example 16. If we have $A = s = 0$, which happens (up to permutations and scaling) for $(a : b : c) = (1 : \omega : \omega^2)$, where ω is a primitive root of 1, we obtain

$$a_n := \sum_{p+q+r=n} \omega^{p-q}\binom{n}{p, q, r}^2 \quad \Rightarrow \quad n^2a_n = 4(4n-9)(4n-3)a_{n-3}. \tag{33}$$

The corresponding differential equation for $\sum a_n t^n$ is

$$(1-64t^3)\Theta^2 + 64t^3\Theta + 20. \tag{34}$$

Note that though the a_n are defined using ω, all terms of the sequence are integers (since $\sum_{i=1}^3 \omega^{ni} = 0$ or 1 depending on $n \mod 3$). The first few terms are:

$$1, 0, 0, 12, 0, 0, 420, 0, 0, 18480, 0, 0, 900900, \ldots$$

(This is sequence A000897 in Sloane's table of integer sequences [S].) Since the recurrence relation is so simple it is easy to verify that the nonzero terms are given by $\binom{4m}{2m,m,m}$, where $m = n/3$. (For example, show that $\binom{4m}{2m,m,m}$ satisfies the above relation; given $a_0 = 1$ the solution is unique.) In other words,

$$\sum_{p+q+r=3m} \omega^{p-q} \binom{3m}{p,q,r}^2 = \binom{4m}{2m,m,m}. \tag{35}$$

4.3 Examples when $N = 4$

Finding explicit equations for the general case for higher N is possible but time consuming, so for $N = 4$ we just give a couple of examples, in cases where the recurrence has fewer terms than in general (when there will be 9 terms), due to relationships between the α_i.

Example 17. For $\{\alpha_i\} = (1, 1, 1, 9)$ the first few a_n are

$$1, 12, 204, 4224, 99324, 2546352, 69359424, 1973611008, 58005903708, \ldots,$$

and the recurrence relation obtained by the above method is

$$\begin{aligned} &(n-1)n^3(10n^2 - 35n + 31)a_n \\ &-4(n-1)(140n^5 - 700n^4 + 1289n^3 - 1104n^2 + 477n - 84)a_{n-1} \\ &+4(1960n^6 - 14700n^5 + 44986n^4 - 71829n^3 + 63127n^2 - 29022n + 5496)a_{n-2} \\ &-1152(n-2)^3(2n-3)(10n^2 - 15n + 6)a_{n-3} \\ &= 0 \end{aligned}$$

The corresponding differential equation is:

$$\begin{aligned} &10(4t-1)(16t-1)(36t-1)\Theta^6 + 45(4608t^3 - 784t^2 + 1)\Theta^5 \\ &+2(378432t^3 - 31172t^2 - 222t - 33)\Theta^4 + (1433088t^3 - 54636t^2 - 148t + 31)\Theta^3 \\ &+4t(371520t^2 - 6217t + 36)\Theta^2 + 24t(33408t^2 - 235t + 3)\Theta + 576t^2(306t - 1). \end{aligned}$$

Example 18. For $\{\alpha_i\} = (1, 1, 1, -3)$ the first few a_n are

$$1, 0, -12, -96, -180, 5760, 70080, 161280, -5144580, -68974080, \ldots,$$

and we obtain

$$\begin{aligned}(n-1)n^3(14n^3-84n^2+165n-107)a_n\\ -4(n-1)^3(2n-5)(14n^3-42n^2+39n-12)a_{n-1}\\ +4(14n^3-84n^2+165n-107)(28n^4-112n^3+163n^2-102n+24)a_{n-2}\\ +192(n-2)^2(28n^5-210n^4+582n^3-737n^2+426n-93)a_{n-3}\\ +2304(n-3)^2(n-2)^2(14n^3-42n^2+39n-12)a_{n-4}\\ =0.\end{aligned}$$

Example 19. For $\{\alpha_i\}=(1,1,-1,-1)$ the first few a_n are

$$1,0,-4,0,156,0,-5440,0,239260,0,-11151504,0,551724096,0,\dots,$$

and we have

$$\begin{aligned}(n-1)n^3(10n^2-55n+76)a_n\\ +4(120n^6-1140n^5+4282n^4-8107n^3+8170n^2-4176n+864)a_{n-2}\\ -1024(n-3)^3(n-2)(10n^2-15n+6)a_{n-4}=0\end{aligned}$$

Example 20. The $N=4$ version of Example 16 is $\{\alpha_i\}=\{1,i,-1,-i\}$. The first few a_n are

$$1,0,0,0,-132,0,0,0,113820,0,0,0,-140078400,0,0,0,201740158620,\dots,$$

and there is a three term recurrence

$$\begin{aligned}0=(n-3)(n-2)(n-1)n^3(48n^4-1032n^3+8276n^2-29347n+38840)a_n\\ +16(6528n^{10}-218688n^9+3180512n^8-26345016n^7+137020240n^6\\ -465036692n^5+1036364052n^4-1486439881n^3\\ +1303139340n^2-627480000n+127008000)a_{n-4}\\ +2^{12}(n-7)^2(n-6)(n-5)^2(n-4)(48n^4-264n^3+500n^2-387n+108)a_{n-8}.\end{aligned}$$

The fact that in this expression the two factors of the form $48n^4+\cdots$ are related by the change of variables $n\mapsto n-4$ leads one to expect that there is probably a recurrence with more terms but coefficients of lower degree.

5 Picard-Fuchs equations in the elliptic curve case

If a sequence satisfies a recurrence relation, then it satisfies infinitely many. So, we do not claim that the recurrence relation obtained by this method give Picard-Fuchs equation. However they could be determined from these relations, once we know (from the geometry) what order the equation should have. For example, for the A_n case, the family of elliptic curves in $\mathbb{P}^2$ is

$$E_t:(X+Y+Z)(aXY+bYZ+cZX)t=XYZ. \tag{36}$$

Aided by computer algebra [Magma], we find that the j-invariant of $E_{1/t}$ is

$$j(E_{1/t}) = \frac{(P+16abct)^3}{(abct)^2 P}, \tag{37}$$

and that for a certain Weierstrass form $y^2 = x^3 - 27c_4 x - 54c_6$

$$\begin{aligned} c_4(E_{1/t}) &= \frac{16(a-t)^4}{a^4(b-c)^4}(P+16abct) &(38)\\ c_6(E_{1/t}) &= \frac{-64(a-t)^6}{a^6(b-c)^6}\sqrt{(P+64abct)}(P-8abct) &(39)\\ \text{where } P &:= \prod_{\varepsilon_1,\varepsilon_2\in\{1,-1\}} \left(\left(\sqrt{a}+\varepsilon_1\sqrt{b}+\varepsilon_2\sqrt{c}\right)^2 - t\right). &(40) \end{aligned}$$

A period of this family given (up to a constant factor) by the series (31), and satisfies a differential equation corresponding to the recurrence relation (32), though this is not the Picard-Fuchs equation, since the degree is too high. However, one can easily show (again a computer algebra package is helpful) that with $R(n)$ as in (32), the expression

$$\begin{aligned} &3(4n-15)R(n) - 2(a+b+c)(4n-11)R(n-1) - s(4n-7)R(n-2),\\ &\text{where } s = a^2+b^2+c^2-2(ab+bc+ca), \end{aligned} \tag{41}$$

is divisible by $(4n-7)(4n-11)(4n-15)$. After dividing we have a recurrence relation

$$\begin{aligned} 0 \;=\; & 3n^2 a_n - A(14n^2-18n+7)a_{n-1}\\ & + \left[A^2(20n^2-56n+41) + s(5n^2-6n-2)\right] a_{n-2}\\ & - \left[2A^3(2n-5)^2 + 4Asn(3n-7) + 12abcF_3F_9\right] a_{n-3}\\ & + \left[A^2 s(4n^2-16n+9) + 8abcAF_{11}F_{13} + s^2(n^2+4n-23)\right] a_{n-4}\\ & + \left[As^2(2n^2-22n+57) + abcsF_{17}F_{19}\right] a_{n-5}, \end{aligned} \tag{42}$$

where $F_p = 4n-p, A = a+b+c$ and $s = a^2+b^2+c^2-2ab-2ac-2cb$. The corresponding Picard-Fuchs differential equation (assuming a,b,c are distinct) is then

$$\Theta^2 + t\sum_{i=1}^{6} \frac{\varepsilon_i}{t-u_i}\Theta + t^2\sum_{i=0}^{6}\frac{\beta_i}{t-u_i}, \tag{43}$$

where $u_0 = 0$, $u_1,\ldots,u_4$ are the values of $1/(\sqrt{a}\pm\sqrt{b}\pm\sqrt{c})^2$, which are real singularities, corresponding to singular elliptic curves, and u_5 and u_6 are apparent singularities, given by the roots of $st^2+2At-3=0$. The

denominators are $\varepsilon_1 = \cdots = \varepsilon_4 = 1$, $\varepsilon_5 = \varepsilon_6 = -1$, $\beta_0 = -a - b - c$, $\beta_5 = -\frac{3}{4u_5}$, $\beta_6 = -\frac{3}{4u_6}$, and

$$\beta_i = \frac{1}{32u_i}\left(\varepsilon_i^1\varepsilon_i^2\sqrt{\frac{1}{abcu_i}}\left(A - \frac{1}{u_i}\right) + 26\right) \text{ for } 1 \le i \le 4, \tag{44}$$

where $u_i = 1/(\sqrt{a} + \varepsilon_i^1\sqrt{b} + \varepsilon_i^2\sqrt{c})^2$.

An alternative method that could be used to find the Picard-Fuchs equation is described by Stienstra and Beukers [SB, §11]. The values of ε_i and β_i could be determined as in [SB], by using knowledge of the solutions, which have the form given in [SB, §11], and the relations given in [Ince, §15.4,16.4].

Example 21. The Picard-Fuchs equation of the family of elliptic curves given by (36) with $a, b, c = 1, 16, 64$ is

$$\begin{aligned} 0 \;\; = \;\; & f'' + \left[\; \tfrac{1}{t} + \tfrac{9}{9t-1} + \tfrac{25}{25t-1} + \tfrac{121}{121t-1} + \tfrac{169}{169t-1} - \tfrac{11}{11t+1} - \tfrac{165}{165t-1} \;\right] f' \\ & +3\left[\begin{array}{l} \frac{-27}{t} + \frac{27\cdot 131}{2^7(9t-1)} + \frac{5^4\cdot 23}{2^7(25t-1)} + \frac{11^4\cdot 53}{2^7(121t-1)} \\ -\frac{13^5}{2^7(169t-1)} + \frac{11^2}{4(11t+1)} - \frac{65^2}{4(65t-1)}. \end{array}\right] f. \end{aligned}$$

6 Further recurrences

We expect that the method of finding recurrences described in this paper should be applicable in many other examples. A simple generalization would be to replace squares by higher powers. The simplest case, when $N = 2$, is the sequence $\{c_n^{a,b,k}\}_n$ with

$$c_n^{a,b,k} := \sum_{p=0}^{k} a^p b^{n-p} \binom{n}{p}^k. \tag{45}$$

In this case we would introduce auxillary terms

$$c_n^{a,b,k,i,j} := \sum_{p+q=n}^{k} a^p b^q p^i q^j \binom{n}{p}^k, \quad k > i, j \ge 0.$$

In the case that $a = b$ we may take $i \ge j$. Similarly to Section 3, we define vector spaces $V_{k,n}^m$ spanned by $c_n^{a,b,k,i,j}$, where $i+j = m$, for $0 \le m \le 2(k-1)$, and we set

$$W_{k,n}^j := V_{k,n}^j \oplus V_{k,n+1}^{k+j}\; 0 \le j < k; \quad W_{k,n}^k := V_{k,n}^k.$$

Now instead of two maps Φ_n^0 and Φ_n^1 (as in (22)), we would have k maps Φ_n^i for $i = 0, \ldots, k-1$, and we would consider a certain composition Ψ_n of these (with image in $W_{k,n-1}^0$ (or $W_{k,n-1}^{k-1}$ in the special case indicated below)), in

order to obtain a relation in one of the W spaces with minimal dimension. (It is helpful to draw a diagram of these spaces, similar to Figure 2.)

In the case where $a = b = 1$ this seems to be more or less the same method as used by [C], and so this method may be viewed as a generalization of his method. Generally $\dim V^m_{k,n} = \min(m+1, 2k-m-1)$, and $\dim W^j_{k,n} = k$, so we would obtain recurrences with $k+1$ terms. But in the case $a = b = 1$, we may identify $c_n^{a,b,k,i,j}$ and $c_n^{a,b,k,j,i}$, so $\dim W^m_{k,n} = \lfloor \frac{k}{2} \rfloor + 1$, unless k is even and $j = k-1$, in which case $\dim W^{k-1}_{k,n} = \frac{k}{2}$, so we obtain recurrances with $\lfloor \frac{k+3}{2} \rfloor$ terms, as in [C].

Example 22. In the case $k = 3$ in (45) we have maps $\Phi^0_n : W^0_n \to W^1_n$, $\Phi^1_n : W^1_n \to W^2_n$, $\Phi^2_n : W^2_n \to W^0_{n-1}$ given by the following three matrices respectively,

$$\begin{pmatrix} \frac{1}{n} & b(1+n)^2 & 0 \\ \frac{1}{n} & 0 & a(n+1)^2 \\ 0 & \frac{1}{(n+1)} & \frac{1}{(n+1)} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{n} & 0 & b(1+n)^2 \\ 0 & \frac{1}{n} & a(1+n)^2 \\ \frac{1}{n} & \frac{1}{n} & 0 \end{pmatrix}, \quad \begin{pmatrix} n^2a & n^2b & 0 \\ 0 & \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & 0 & \frac{1}{n} \end{pmatrix}$$

with respect to the bases $\{a_n^{0,0}, a_{n+1}^{1,2}, a_{n+1}^{2,1}\}$ for W^0_n; $\{a_n^{1,0}, a_n^{0,1}, a_{n+1}^{2,2}\}$ for W^1_n; $\{a_n^{2,1}, a_n^{0,2}, a_n^{1,1}\}$ for W^2_n, and $\{a_{n-1}^{0,0}, a_n^{1,2}, a_n^{2,1}\}$ for W^0_{n-1}. We set $\Psi_n = \Phi^2_n\Phi^1_n\Phi^0_n$, and consider $\Psi_{n-2}\Psi_{n-1}\Psi_n(a_n), \Psi_{n-2}\Psi_{n-1}(a_{n-1}), \Psi_{n-2}(a_{n-2})$ and a_{n-3}, given in W^0_{n-2} (with basis $\{a_{n-2}^{0,0}, a_{n-1}^{1,2}, a_{n-1}^{2,1}\}$) by the columns of the following matrix:

$$\begin{pmatrix} \frac{3abA(3(n-2)(n-1)^2(9n-4)+4n)}{(n-1)^2n^2} + A^3 & \frac{6(n-2)(3n-5)ab}{(n-1)^2} + A^2 & A & 1 \\ \frac{*}{(n-2)^3(n-1)^2n^2} & \frac{3(n-1)(2n-3)b+3a(5n^2-16n+13)}{(n-2)^3(n-1)^2} & \frac{3}{(n-1)^3} & 0 \\ \frac{*}{(n-2)^3(n-1)^2n^2} & \frac{3(n-1)(2n-3)a+3b(5n^2-16n+13)}{(n-2)^3(n-1)^2} & \frac{3}{(n-1)^3} & 0 \end{pmatrix},$$

where $A = a + b$, and $*$ denoted unenlightening polynomials of degree 4 in n and degreee 2 in a and b. Note that the determinant of the matrix consisting of the last three columns is $\frac{9(a+b)^2(a-b)(3n-2)}{n^2(n-1)^2(n-2)^3}$, and so in the case $a = \pm b$ there are recurrences with at most 3 terms (these can be found for example in [W], so we do not give them here). Otherwise we expect recurrences with 4 terms. In general by finding the kernel of this matrix we find that the sequence $c_n := c_n^{a,b,3}$ satisfies a relation

$$\begin{aligned} 0 &= 3n^2(3n-5)c_n - (27n^3 - 72n^2 + 51n - 12)(a+b)c_{n-1} \\ &- \left[(a+b)^2 + (3n-5)\Big((a+b)^2 + (9ab - (a+b)^2)(3n-4)(3n-2)\Big)\right] c_{n-2} \\ &- 3(a+b)^3(n-2)^2(3n-2)c_{n-3}. \end{aligned} \tag{46}$$

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